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Probability Theory and Related Fields

Mimicking the One-Dimensional Marginal Distributions of Processes Having an Ito Differential

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Summary. Let $\xi(t)$ be a stochastic process starting from 0 with Ito differential

$$d\xi(t)=\delta(t,\omega)\,dW(t)+\beta(t,\omega)\,dt,$$

where $(W(t),\mathfrak{F}_t)$ is a Wiener process, δ and β are bounded $\mathfrak{F}_t$-nonanticipative processes such that $\delta\delta^T$ is uniformly positive definite. Then it is proved that there exists a stochastic differential equation

$$dx(t)=\sigma(t,x(t))\,dW(t)+b(t,x(t))\,dt$$

with non-random coefficients which admits a weak solution $x(t)$ having the same one-dimensional probability distribution as $\xi(t)$ for every t. The coefficients σ and b have a simple interpretation:

$$\sigma(t,x)=(E(\delta\delta^T(t)\,|\,\xi(t)=x))^{\frac{1}{2}},\ b(t,x)=E(\beta(t)\,|\,\xi(t)=x).$$

1. Introduction

The subject of this paper is the construction of simple stochastic differential equations whose solutions mimic certain features of the behaviour of the solutions of more complicated equations. The prototype for investigations of this kind, and the immediate inspiration for the work presented here, is a recent result of N.V. Krylov [3], which is described in some detail below. Our first task will be the filling of a small gap in Krylov's proof of a crucial lemma. We then prove an analogous lemma in another setting.

Let $\xi(t)$ be a stochastic process of the form

$$\xi(t)=\int_0^t\delta(s,\omega)\,dW(s)+\int_0^t\beta(s,\omega)\,ds, \tag{1.0}$$

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where $(W(s), \mathfrak{F}_s)$ is a k-dimensional Wiener process on a probability space $(\Omega, \mathfrak{F}, P)$ equipped with a stochastic basis $(\mathfrak{F}_s)_{s \geqq 0}$ and δ, β are bounded measurable $\mathfrak{F}_s$-adapted processes taking values in $\mathbb{R}^{n\times k}$ and $\mathbb{R}^n$ respectively. The path-wise behaviour of $\xi(t)$ may depend, via the random coefficients $\delta(s,\omega)$ and $\beta(s,\omega)$, in an extremely complicated way on the past $\mathfrak{F}_t$. But suppose we are interested in an aspect of the behaviour of ξ which depends only on, say, the position in the state-space: for instance, the amount of time that the process spends in the vicinity of a given point. Then one might hope to find a stochastic process $x(t)$ mimicking this feature (in a sense to be made precise below) and satisfying a "simpler" stochastic differential equation of the form

$$x(t)=\int_0^t \sigma(x(t))\,dW(t)+\int_0^t b(x(t))\,dt,$$

where the coefficients σ and b are now non-random, and depend only on the position in the state space.

To formulate Krylov's result, consider the Green measure, defined by

$$\mu(\Gamma)=E\int_0^\infty \chi_\Gamma(\xi(t))\exp\left(-\int_0^t \gamma(s,\omega)\,ds\right)dt$$

(χ_Γ denotes the indicator-function of the set Γ) for every Borel set $\Gamma\subset\mathbb{R}^n$, where γ is a non-negative $\mathfrak{F}_t$-adapted stochastic process, the killing rate. (If $\xi(\omega)$ is killed on the time interval $(t,t+\Delta t)$ with probability $\gamma(t,\omega)\Delta t+o(\Delta t)$, then $\exp\left(-\int_0^t \gamma(s,\omega)\,ds\right)$ is the probability of being alive at time t.) Recall that one says that the stochastic differential equation

$$dx(t)=f(t,x(t))\,dW(t)+g(t,x(t))\,dt; \qquad x(0)=0$$

has a weak solution, say $\bar{x}(t)$, if there exists a probability space $(\bar{\Omega}, \bar{\mathfrak{F}}, \bar{P})$ and a Wiener process $(\bar{w}(t), \bar{\mathfrak{F}}_t)$ on it, such that $\bar{x}(t)$ is an $\bar{\mathfrak{F}}_t$-adapted stochastic process which satisfies the equation

$$\bar{x}(t)=\int_0^t f(s,\bar{x}(s))\,d\bar{w}(s)+\int_0^t g(s,\bar{x}(s))\,ds$$

(see e.g. [6]).

We can now formulate Krylov's theorem as follows: Consider a process of the form (1.0). Let $\gamma(t,\omega)$ be fixed. Suppose that for all $t\geqq 0$ and $\omega\in\Omega$

$$\delta\delta^T(t,\omega)\geqq\lambda_1 I, \qquad \gamma(t,\omega)\geqq\lambda_2, \tag{1.1}$$

where δ^T is the transpose of the matrix δ, I is the $n\times n$ identity matrix and λ_1, λ_2 are positive constants. Then one can define non-random functions σ: $\mathbb{R}^n\to\mathbb{R}^{n\times n}$, b: $\mathbb{R}^n\to\mathbb{R}^n$ and c: $\mathbb{R}^n\to\mathbb{R}$, such that the time-homogeneous stochastic equation

$$dx(t)=\sigma(x(t))\,dW(t)+b(x(t))\,dt, \qquad x(0)=0 \tag{1.2}$$

has a weak solution, having the same Green measure (with killing rate $c(x(t))$) as ξ (with the killing rate $\gamma(t)$).

Krylov's result makes it natural to ask whether it is possible to construct simple processes which mimic other aspects of the behaviour of $\xi(t)$. Following a suggestion of Krylov, we show that this is possible for all the one-dimensional marginal distributions of ξ, that is, that one can construct a process x satisfying a simple stochastic differential equation such that the distributions of $\xi(t)$ and $x(t)$ are identical for all t. Naturally the coefficients in this equation will be functions of (t,x):

$$dx(t)=a^{\frac{1}{2}}(t,x(t))\,dW(t)+b(t,x(t))\,dt \tag{1.3}$$

($a^{\frac{1}{2}}$ denotes the positive definite square root of the positive definite matrix a). The coefficients in this equation, unlike those in (1.2) have a simple intuitive interpretation:

$$a(t,x)=E(\delta(t)\,\delta^T(t)\,|\,\xi(t)=x)$$

$$b(t,x)=E(\beta(t)\,|\,\xi(t)=x).$$

Note that our theorem does not imply Krylov's result, since the Eq. (1.3) is not time-homogeneous. Indeed, it might seem, at first glance, that our result follows from Krylov's by considering the time-space process $(t,\xi(t))$. But this is not so: Krylov's theorem does not apply to this process because the diffusion coefficient does not satisfy the regularity condition (1.1).

To get an idea of the method of proof, consider the special case in which $\beta=0$ and all the processes are scalar valued. Choose a smooth function $u(t,x)$ which is identically 0 for large $t+|x|$ (i.e. $u\in C_0^\infty([0,\infty)\times\mathbb{R})$). Apply Ito's formula to $u(t,\xi(t))$, where $\xi(t)=\int_0^t\delta(s,\omega)\,dW(s)$:

$$du(t,\xi(t))=\frac{\partial u}{\partial t}(t,\xi(t))\,dt+\frac{\partial u}{\partial x}(t,\xi(t))\,d\xi(t)+\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,\xi(t))\,\delta^2(t)\,dt.$$

Taking expectations we get

$$\begin{aligned}Eu(T,\xi(T))-u(0,0)&=E\int_0^T\left(\frac{\partial}{\partial t}u(t,\xi(t))+\frac{1}{2}\delta^2(t)\frac{\partial^2}{\partial x^2}u(t,\xi(t))\right)dt\\&=\int_0^T E\left(\left\{\frac{\partial}{\partial t}+\frac{1}{2}E(\delta^2(t)\,|\,\xi(t))\frac{\partial^2}{\partial x^2}\right\}u(t,\xi(t))\right)dt\end{aligned}$$

for every $T\geqq 0$. This gives us for every $u\in C_0^\infty([0,\infty)\times\mathbb{R})$

$$\int_0^\infty\int_{-\infty}^\infty u(t,x)\,\nu(dt,dx)=-\int_0^\infty\int_{-\infty}^\infty\left(\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\right)u(t,x)\,d\mu_t(x)\,dt \tag{1.4}$$

with $\mu_t(dx):=P(\xi(t)\in dx)$, $\sigma(t,x):=(E(\delta^2(t)\,|\,\xi(t)=x))^{\frac{1}{2}}$ and Dirac measure ν, concentrated at $(0,0)\in\mathbb{R}^{n+1}$.

The relation (1.4) can be taken as an equation satisfied by the distributions $P(\xi(t)\in dx)$ of the original process. Consider on the other hand a weak solution of the stochastic differential equation

$$dx(t)=\sigma(t,x(t))\,dW(t),\qquad x(0)=0. \tag{1.5}$$

It is clear by Ito's formula that the one-dimensional distributions $\bar{P}(x(t)\in dx)$ satisfy (1.4) for every $u\in C_0^\infty([0,\infty)\times\mathbb{R})$ as well.

This consideration suggests how to mimic the process $\xi(t)$ with respect to its one-dimensional distributions. Since σ^2 can be chosen uniformly positive definite and bounded, a well-known theorem guarantees that (1.5) has indeed a weak solution. If we could prove that the Eq. (1.4) suffices to identify the distributions $P(\xi(t)\in dx)$ then our theorem would be proved. Unfortunately a uniqueness theorem of this kind has only been shown for smooth functions $\sigma(t,x)$. Following Krylov we circumvent this difficulty as follows. Starting from the Eq. (1.4) we choose approximations of σ by smooth coefficients σ_ε with random initial conditions such that the Green measures of the corresponding processes $(u_\varepsilon(t),x_\varepsilon(t))$ converge weakly to the Green measure of $(t,\xi(t))$, i.e., to $P(\xi(t)\in dx)\,dt$. On the other hand we see that the distribution of $(u_\varepsilon,x_\varepsilon)$ on $C[0,T]$ converges weakly to the distribution of the process $(t,x(t))$, provided $\sigma_\varepsilon\to\sigma$, where $x(t)$ is a weak solution of the Eq. (1.5). Hence we get our result by noting that the coincidence of the Green measures of the processes $(t,\xi(t))$ and $(t,x(t))$ implies the coincidence of the one-dimensional distributions of the processes $\xi(t)$ and $x(t)$.

The main step in the proof is this approximation procedure. It forms the content of our Lemma 2.2, which will be a time-dependent version of Krylov's lemma, restated below as Lemma 2.1.

Lemma 2.2 deals with a more general situation: Instead of the measures ν and $d\mu_t(x)\,dt$ from (1.4) we consider arbitrary measures ν and μ on $[0,\infty)\times\mathbb{R}^n$, and instead of the Eq. (1.4) we start with an inequality of the form

$$\int_0^\infty\int_{\mathbb{R}^n} u(t,x)\,\nu(dt,dx)\geqq -\int_0^\infty\int_{\mathbb{R}^n} Lu(t,x)\,\mu(dt,dx)$$

valid for every non-negative function $u\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$, where L is a second order partial differential operator of parabolic type.

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2. Notations and the Fundamental Lemmas

Let S_n^+ denote the set of all symmetric positive semidefinite $n\times n$ matrices. With $a:=(a^{ij})$, $b:=(b^i)$ and c bounded Borel measurable functions on $\mathbb{R}^n$ (taking values in S_n^+, $\mathbb{R}^n$ and $\mathbb{R}$ respectively) we associate the differential operator

$$L=\frac{1}{2}\sum_{i,j}a^{ij}(x)\frac{\partial^2}{\partial x^i\partial x^j}+\sum_i b^i(x)\frac{\partial}{\partial x^i}-c(x).$$

In the situation in which we will be interested we are given two non-negative finite measures ν and μ with the following property

$$\int_{\mathbb{R}^n} u(x)\,\nu(dx)\geqq -\int_{\mathbb{R}^n} Lu(x)\,\mu(dx) \tag{2.1}$$

for every non-negative function $u\in C_0^\infty(\mathbb{R}^n)$.

(If X is a subset of a Euclidean space then $C_0^\infty(X)$ denote the set of all infinitely differentiable functions $u: X \to \mathbb{R}$ having compact support.)

We shall make use of the following smoothing devices. We fix a non-negative function $k \in C^\infty(\mathbb{R}^n)$ such that $\int k(x)\,dx = 1$ and $k(-x) = k(x)$ for all x. Put $k^\varepsilon(x) = \varepsilon^{-n} k(\varepsilon^{-1} x)$ for every $\varepsilon > 0$.

For bounded measurable function g we define the measure $g\mu$ by

$$(g\mu)(\Gamma) = \int \chi_\Gamma(x)\, g(x)\, \mu(dx).$$

Convoluting this measure with the measure $k^{(\varepsilon)}(x)\,dx$ we define the measure $(g\mu)^{(\varepsilon)}(x)\,dx$, where the density function (with respect to the Lebesgue measure) is given by

$$(g\mu)^{(\varepsilon)}(x) = \int k^\varepsilon(x-y)\, g(y)\, \mu(dy).$$

Suppose that $\mu^{(\varepsilon)}(x)$ is positive on $\mathbb{R}^n$ for every ε. Then the measure $(g\mu)^{(\varepsilon)}\,dx$ has a smooth Radon-Nikodym derivative

$$g_{(\varepsilon)} := \frac{(g\mu)^{(\varepsilon)}}{\mu^{(\varepsilon)}} \tag{2.2}$$

with respect to measure $\mu^{(\varepsilon)}\,dx$ for every bounded measurable g.

We smooth the coefficients of L by the formula (2.2) and together with L we consider the operators

$$L_\varepsilon = \frac{1}{2} \sum_{i,j} a^{ij}_{(\varepsilon)}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_i b^i_{(\varepsilon)}(x) \frac{\partial}{\partial x^i} - c_{(\varepsilon)}(x)$$

and the time-homogeneous stochastic differential equations

$$dx(t) = (a_{(\varepsilon)})^{\frac{1}{2}}(x(t))\, dW(t) + b_{(\varepsilon)}(x(t))\, dt \tag{2.3}$$

defined on a probability space $(\Omega, \mathfrak{F}, P)$ equipped with a filtration $(\mathfrak{F}_t)_{t \geqq 0}$ and a Wiener process $(W(t), \mathfrak{F}_t)$.

(By $(a_{(\varepsilon)})^{\frac{1}{2}}$ we denote the symmetric positive semidefinite square root of the matrix $a_{(\varepsilon)}$.)

We denote by x^x_ε a solution of this equation with initial condition $x(0) = x \in \mathbb{R}^n$. For a function g we denote by $g^{(\varepsilon)}$ the convolution $g * k^\varepsilon$.

Now we formulate the fundamental lemma in the time-homogeneous case.

Lemma 2.1 (Krylov [3]). *Let the differential operator L and the measures ν and μ be as above, in particular satisfying 2.1. Assume $c(x) \geqq p > 0$ for every $x \in \mathbb{R}^n$. Define for every positive constant λ and $\varepsilon > 0$ the operators $\mathfrak{R}^\varepsilon_\lambda$:*

$$\mathfrak{R}^\varepsilon_\lambda f(x) = E \int_0^\infty f(x^x_\varepsilon(t)) \exp\left(-\int_0^t c_{(\varepsilon)}(x^x_\varepsilon(s))\, ds - \lambda t\right) dt.$$

Then for every non-negative function $f \in C_0^\infty(\mathbb{R}^n)$ the functions $\mathfrak{R}^\varepsilon_0 f(x)$ are continuous in x and

$$\int_{\mathbb{R}^n} (\mathfrak{R}^\varepsilon_0 f)^{(\varepsilon)}(x)\, \nu(dx) \geqq \int_{\mathbb{R}^n} f^{(\varepsilon)}(x)\, \mu(dx), \tag{2.4}$$

in consequence

$$\liminf_{\varepsilon \downarrow 0} \int (\mathfrak{R}_0^\varepsilon f)^{(\varepsilon)}(x)\, \nu(dx) \geqq \int f(x)\, \mu(dx). \tag{2.5}$$

If the inequality in (2.1) *is reversed then it is also reversed in* (2.4) *and* (2.5), *where* lim inf *in* (2.5) *is replaced by* lim sup.

Our definitions in the time-dependent case are straight-forward generalizations of the above. Thus we consider the operator

$$L = \frac{\partial}{\partial t} + \frac{1}{2} a^{ij}(t,x) \frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x) \frac{\partial}{\partial x^i} - c(t,x),$$

where $a=(a^{ij})$, $b=(b^i)$ and c are bounded measurable functions defined on $[0,\infty)\times\mathbb{R}^n$ and take values in the same sets as before.

We suppose that for every $(t,x)\in[0,\infty)\times\mathbb{R}^n$

$$\sum_{i,j} a^{ij}(t,x)\, z^i z^j \geqq \lambda |z|^2 \tag{2.6}$$

for every $z=(z^i)\in\mathbb{R}^n$, where λ is some fixed positive constant and $|z|$ denotes the norm of z in $\mathbb{R}^n$.

The measures ν and μ are supposed to be defined on the Borel sets of $[0,\infty)\times\mathbb{R}^n$. Instead of the kernel k we consider the kernels $h(t,x)=\psi(t)\,k(x)$ and $h^*(t,x)=h(-t,-x)=\psi(-t)\,k(x)$, where $\psi\in C_0^\infty(\mathbb{R})$ such that $\int\psi(t)\,dt=1$, $\psi(t)>0$ on $(-1,0)$ and $\psi(t)=0$ elsewhere. We set $h^\varepsilon(t,x)=\varepsilon^{-(n+1)}h(\varepsilon^{-1}t,\varepsilon^{-1}x)$ and $h^{*\varepsilon}(t,x)=h^\varepsilon(-t,-x)$, and denote by $m^{(\varepsilon)}$ and $m^{(*\varepsilon)}$ the convolutions $(\chi_{t\geqq 0}m)*h^\varepsilon$ and $(\chi_{t\geqq 0}m)*h^{*\varepsilon}$ respectively, where m is a measure or a function on $\mathbb{R}^{n+1}$. The smoothing is defined by

$$f_{(\varepsilon)} = (f\mu)^{(*\varepsilon)} (\mu^{(*\varepsilon)})^{-1} \tag{2.7}$$

for functions f defined on $\mathbb{R}^{n+1}$.

We consider the operators

$$L_\varepsilon = \frac{\partial}{\partial t} + \frac{1}{2}\sum_{i,j} a_{(\varepsilon)}^{ij}(t,x) \frac{\partial}{\partial x^i \partial x^j} + \sum_j b_{(\varepsilon)}^i(t,x) \frac{\partial}{\partial x^i} - c_{(\varepsilon)}(t,x) \tag{2.8}$$

and the stochastic differential equations

$$dx(t) = (a_{(\varepsilon)})^{\frac{1}{2}}(t+s, x(t))\, dW(t) + b_{(\varepsilon)}(t+s, x(t))\, dt \tag{2.9}$$

for every $s\geqq 0$, where the coefficients $a_{(\varepsilon)}$, $b_{(\varepsilon)}$ and $c_{(\varepsilon)}$ are defined by the formula (2.7) from a, b, and c respectively.

Our fundamental lemma is formulated as follows.

Lemma 2.2. *Assume that* $\infty > \mu^{(*\varepsilon)}(t,x) > 0$ *on* $(0,\infty)\times\mathbb{R}^n$ *for every* $\varepsilon>0$. *Suppose that*

$$\int_{[0,\infty)\times\mathbb{R}^n} u(t,x)\, \nu(dt,dx) \geqq - \int_{[0,\infty)\times\mathbb{R}^n} Lu(t,x)\, \mu(dt,dx) \tag{2.10}$$

for every $u\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$.

($C_0^\infty([0,\infty)\times\mathbb{R}^n)$ denotes the set of infinitely differentiable functions f, having compact support, i.e., $f(t,x)=0$ if $t+|x|$ is large enough.)

Let x_ε^{sx} be a solution of the Eq. (2.9) *with initial condition $x_\varepsilon^{sx}(0)=x\in\mathbb{R}^n$, and let*

$$\mathfrak{R}^\varepsilon f(s,x)=E\int_0^\infty f(s+t,x_\varepsilon^{sx}(t))\exp\left(-\int_0^t c_{(\varepsilon)}(s+r,x_\varepsilon^{sx}(r)\,dr\right)dt.$$

Then for every non-negative $f\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$

$$\int_{[0,\infty)\times\mathbb{R}^n}(\mathfrak{R}^\varepsilon f)^{(\varepsilon)}\,\nu(dt,dx)\geqq\int_{[0,\infty)\times\mathbb{R}^n}f^{(\varepsilon)}\,\mu(dt,dx)\tag{2.11}$$

for every $\varepsilon>0$, and thus

$$\liminf_{\varepsilon\downarrow 0}\int(\mathfrak{R}^\varepsilon f)^{(\varepsilon)}\,\nu(dt,dx)\geqq\int f\,\mu(dt,dx).\tag{2.12}$$

If inequality in (2.10) *is reversed then it is also reversed in* (2.11) *and* (2.12) *with* lim sup *in* (2.12) *instead of* lim inf.

3. The Proofs of the Fundamental Lemmas

Krylov's proof of Lemma 2.1 is incomplete because the coefficients $a_{(\varepsilon)}$, $b_{(\varepsilon)}$ and $c_{(\varepsilon)}$ are assumed to have bounded derivatives in x up to the second order, which does not hold in the case of general kernels k.

In this section we show that it is easy to get around this difficulty by approximating the kernel k and using a limit theorem from [2]. We use the same approximation procedure in the proof of Lemma 2.2 as well. In both cases we use a special kernel

$$\rho(x)=(1+|x|^2)^{-n}\Big(\int_{\mathbb{R}^n}(1+|x|^2)^{-n}\,dx\Big)^{-1},$$

and we make use of the fact that ρ has the property

$$\left|\frac{\partial}{\partial x^i}\rho(x)\right|\leqq K\rho(x),\qquad\left|\frac{\partial^2}{\partial x^i\,\partial x^j}\rho(x)\right|\leqq K\rho(x)\tag{3.1}$$

for every $i,j=1,2,\dots,n$, where K is some positive constant.

The Proof of Lemma 2.1

Step 1. Assume, in addition, that for every $\varepsilon>0$ the kernel k^ε has the property (3.1). This implies that $a_{(\varepsilon)}^{ij}$, $b_{(\varepsilon)}^i$ and $c_{(\varepsilon)}$ belong to $C_b^2(\mathbb{R}^n)$. ($C_b^m(\mathbb{R}^n)$ is the set of functions $f\colon\mathbb{R}^n\to\mathbb{R}$ possessing bounded continuous derivatives of order up to and including m.) In this case Krylov's proof goes through without change. For sake of completeness we repeat his proof now.

Since $a_{(\varepsilon)}(x)$ is non-negative definite for every $x\in\mathbb{R}^n$ and $a_{(\varepsilon)}\in C_b^2(\mathbb{R}^n)$, one knows by a theorem of Freidlin [1] that $(a_{(\varepsilon)})^{\frac{1}{2}}$ satisfies a Lipschitz condition on $\mathbb{R}^n$. It is easy to see that $a^{ij}_{(\varepsilon)}$, $b^i_{(\varepsilon)}$ and $c_{(\varepsilon)}$ are bounded, uniformly in ε. Consequently, by the classical existence and uniqueness theorem of Ito, Eq. (2.3) has a unique (strong) solution x^x_ε for every initial condition $x^x_\varepsilon(0)=x\in\mathbb{R}^n$. Moreover it is well-known that $x^x_\varepsilon(t)$ depends continuously (in probability) on the initial value x and that x^x_ε is a Markov process. Thus we have

$$\begin{aligned}\mathfrak{R}^\varepsilon_\vartheta &= \mathfrak{R}^\varepsilon_\kappa + (\kappa-\vartheta)\,\mathfrak{R}^\varepsilon_\kappa\,\mathfrak{R}^\varepsilon_\vartheta \\ \mathfrak{R}^\varepsilon_\vartheta &= \sum_{n=1}^{\infty} (\kappa-\vartheta)^{n-1}(\mathfrak{R}^\varepsilon_\kappa)^n\end{aligned} \tag{3.2}$$

for $\kappa>-p$, $\vartheta>-p$ and $|\kappa-\vartheta|<p+\kappa$, where the series converges in the Banach space of linear operators mapping $C_b(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$. (It is easy to see that $|\mathfrak{R}^\varepsilon_\kappa|\leqq(p+\kappa)^{-1}$.) From [7] it is known that there exists $\lambda_\varepsilon>0$ such that for every $\lambda\geqq\lambda_\varepsilon$ and $f\in C_0^\infty(\mathbb{R}^n)$ the equation $\lambda v-L_\varepsilon v=f$ admits a solution $v_\varepsilon\in C_b^2(\mathfrak{R}^n)$. By the Ito formula $v_\varepsilon=\mathfrak{R}^\varepsilon_\lambda f$. One easily sees that inequality (2.1) is valid also for non-negative functions $\bar{u}\in C^\infty(\mathbb{R}^n)$, which are bounded and have bounded first and second derivatives. In order to show this, let $w\in C_0^\infty(\mathbb{R}^n)$ such that $0\leqq w\leqq 1$ on $\mathbb{R}^n$ and $w(x)=1$ if $|x|\leqq 1$. Then replacing $u(x)$ in (2.1) by $\bar{u}(x)\,h(rx)$ for $r>0$ and taking $r\to 0$ we get

$$\int_{\mathbb{R}^n} \bar{u}(x)\,v(dx)\geqq -\int L\bar{u}(x)\,\mu(dx) \tag{3.3}$$

by the Lebesgue theorem. Now replacing $\bar{u}$ in (3.3) by $(v_\varepsilon)^{(\varepsilon)}$ and then using the Fubini theorem and the symmetry of the kernel k^ε we obtain

$$\int v_\varepsilon(x)\,v^{(\varepsilon)}(x)\,dx\geqq -\int L_\varepsilon v_\varepsilon(x)\,\mu^{(\varepsilon)}(x)\,dx.$$

Hence, since $-L_\varepsilon v_\varepsilon=-\lambda v_\varepsilon+f$ and $v_\varepsilon=\mathfrak{R}^\varepsilon_\lambda f$, we get

$$\int \mathfrak{R}^\varepsilon_\lambda f\,v^{(\varepsilon)}\,dx\geqq\int(-\lambda\,\mathfrak{R}^\varepsilon_\lambda f+f)\,\mu^{(\varepsilon)}\,dx \tag{3.4}$$

for every non-negative $f\in C_0^\infty(\mathbb{R}^n)$, $\lambda\geqq\lambda_\varepsilon$ and $\varepsilon>0$. Now we show that inequality (3.4) holds for $\lambda=0$ as well.

First we extend inequality (3.4) to bounded non-negative functions f by the Lebesgue convergence theorem. Then using (3.2) and (3.4) we see that for $\lambda>\lambda_\varepsilon$

$$\begin{aligned}\int \mathfrak{R}^\varepsilon_0 f(x)\,v^{(\varepsilon)}\,dx &= \sum_{n=1}^{\infty}\lambda^{n-1}\int(\mathfrak{R}^\varepsilon_\lambda)^n f(x)\,v^{(\varepsilon)}(x)\,dx \\ &\geqq \sum_{n=1}^{\infty}\lambda^{n-1}\int\{-\lambda(\mathfrak{R}^\varepsilon_\lambda)^n f(x)+(\mathfrak{R}^\varepsilon_\lambda)^{n-1}f(x)\}\,\mu^{(\varepsilon)}\,dx \\ &= \int f(x)\,\mu^{(\varepsilon)}(x)\,dx.\end{aligned}$$

Now using the Fubini theorem and the symmetry of k again we get the inequality

$$\int(\mathfrak{R}^\varepsilon_0 f)^{(\varepsilon)}\,v(dx)\geqq\int f^{(\varepsilon)}\,\mu(dx).$$

Step 2. Define $k^{\varepsilon r}=k^\varepsilon * \rho^r$. Note that the kernels $k^{\varepsilon r}$ have the property (3.1). Thus, by virtue of what we have just proved in step 1,

$$\int (\mathfrak{R}_0^{\varepsilon r} f)^{(\varepsilon r)}\, \nu(dx) \geqq \int f^{(\varepsilon r)}\, \mu(dx) \tag{3.5}$$

for every ε, $r>0$ and non-negative $f \in C_0^\infty(\mathbb{R}^n)$, where the double indices εr and (εr) indicate that the kernel $k^{\varepsilon r}$ is used (instead of the kernel k^ε) in the corresponding definitions. Next we let $r \downarrow 0$ in (3.5) using the following

Proposition 3.1. *Let $x_{\varepsilon r}^x$ be the solution of the equation*

$$dx(t) = (a_{(\varepsilon r)})^{\frac{1}{2}}(x(t)\, dW(t) + b_{(\varepsilon r)}(x(t))\, dt$$

with initial condition $x_{(\varepsilon r)}^x(0) = x$, where $a_{(\varepsilon r)}$, $b_{(\varepsilon r)}$ are defined like $a_{(\varepsilon)}$ and $b_{(\varepsilon)}$ but using the kernel $k^{\varepsilon r}$ instead of k^ε. Then for $r \downarrow 0$ we have $x_{\varepsilon r}(t) \to x_\varepsilon(t)$ in probability, uniformly in t on every finite interval.

Proof. It is easy to see that for $r \downarrow 0$

$$(a_{(\varepsilon r)})^{\frac{1}{2}}(x) \to (a_{(\varepsilon)})^{\frac{1}{2}}(x) \quad \text{and} \quad b_{(\varepsilon r)}(x) \to b_{(\varepsilon)}(x)$$

for $x \in \mathbb{R}^n$, uniformly on every compact set of $\mathbb{R}^n$. Moreover $(a_{(\varepsilon)})^{\frac{1}{2}}$ and $(b_{(\varepsilon)})^{\frac{1}{2}}$ are bounded and they satisfy a Lipschitz condition on $\{x : |x| \leqq R\}$ for every R. Thus our proposition follows immediately from Theorem 3 of [2].

The proposition implies that

$$\mathfrak{R}_0^{\varepsilon r} f(x) \to \mathfrak{R}_0^\varepsilon f(x) \quad \text{as } r \downarrow 0, \qquad \text{for every } x \in \mathbb{R}^n.$$

Thus taking $r \downarrow 0$ in (3.5), by the Lebesgue theorem we get

$$\int (\mathfrak{R}_0^\varepsilon f)^{(\varepsilon)}\, \nu(dx) \geqq \int f^{(\varepsilon)}\, \mu(dx). \tag{3.6}$$

Starting with the reversed inequality in (2.1) we can prove the opposite inequality in (3.6). The rest of the lemma is evident.

The Proof of Lemma 2.2

Step 1. Suppose that h has the property

$$\begin{aligned} \left| \frac{\partial}{\partial x^i} h(t,x) \right| &\leqq K h(t,x) \\ \left| \frac{\partial}{\partial x^i \partial x^j} h(t,x) \right| &\leqq K h(t,x) \end{aligned} \tag{3.7}$$

for all $i, j = 1, 2, \ldots, n$. Then $a_{(\varepsilon)}$, $b_{(\varepsilon)}$ and $c_{(\varepsilon)}$ have bounded continuous spatial derivatives up to the second order on $(0,\infty) \times \mathbb{R}^n$ for every $\varepsilon > 0$. Consequently $(a_{(\varepsilon)})^{\frac{1}{2}}$ and $b_{(\varepsilon)}$ satisfy a Lipschitz condition in x on $(0,\infty) \times \mathbb{R}^n$. It is easy to see that $a_{(\varepsilon)}$, $b_{(\varepsilon)}$ and $c_{(\varepsilon)}$ are bounded continuous functions on $(0,\infty) \times \mathbb{R}^n$. Note that $a_{(\varepsilon)}$, $b_{(\varepsilon)}$ and $c_{(\varepsilon)}$ are not defined if $t=0$. But since it suffices that the conditions of the existence and uniqueneness theorem be satisfied at every x for almost all t only (see e.g. [2]), it follows that Eq. (2.3) has a unique (strong) solution x_ε^{sx} for every $\varepsilon > 0$.

Now we need the following

Proposition 3.2. *For every non-negative $f \in C_0^\infty([0,\infty)\times\mathbb{R}^n)$ the equation $L_\varepsilon u+f=0$ has a unique solution $u_\varepsilon \in C_b^{1,2}((0,\infty)\times\mathbb{R}^n)$. ($C_b^{m,1}(X)$ for $X\subset[0,\infty)\times\mathbb{R}^n$ is the set of functions f: $X\to\mathbb{R}$ such that f has m bounded continuous time derivatives and bounded continuous spatial derivatives at order less than or equal to 1 on X.) This solution is also non-negative.*

Proof. This follows immediately from a theorem of Krylov ([5] Th. II.9.10), which can be formulated for our purposes as follows:

Let $\bar{\sigma}=(\bar{\sigma}^{ij})$, $\bar{b}=(b^i)$, $\bar{c}$ and $\bar{f}$ be bounded continuous functions on $[r,\infty)\times\mathbb{R}^n$ taking values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$ and $\mathbb{R}$ respectively. Suppose that their derivatives in x up to second order are bounded and continuous functions on $[r,\infty)\times\mathbb{R}^n$. Assume moreover that $\bar{f}(t,x)=0$ if $t\geqq T$ for some fixed $T\in[r,\infty)$.

Set $\bar{a}=\frac{1}{2}\bar{\sigma}\bar{\sigma}^T$. Then the equation

$$\frac{\partial}{\partial t}u+\sum_{i,j}\bar{a}^{ij}\frac{\partial^2}{\partial x^i\partial x^j}u+\sum_j\bar{b}_i\frac{\partial}{\partial x^i}\bar{u}-c\bar{u}+\bar{f}=0$$

has a unique solution $\bar{u}\in C_b^{1,2}([r,\infty)\times\mathbb{R}^n)$.

Moreover, this solution is non-negative and

$$\sup_{(t,x)\in[r,\infty)\times\mathbb{R}^n}\left(|\bar{u}|+\sum_i\left|\frac{\partial}{\partial x^i}\bar{u}\right|+\sum_{i,j}\left|\frac{\partial^2}{\partial x^i\partial x^j}\bar{u}\right|\right)\leqq K,$$

where the constant K depends only on n, T, on the suprema of the functions $\bar{\sigma}$, $\bar{b}$, $\bar{c}$, $\bar{f}$, and on the suprema of their spatial derivatives up to the second order. Applying this theorem to the equation $L_\varepsilon u+f=0$ in the domains $[r_n,\infty)\times\mathbb{R}^n$ for $0<r_n\downarrow 0$, we obtain the desired result.

From inequality (2.10) it is easy to get that

$$\int_{[0,\infty)\times\mathbb{R}^n} u^{(\varepsilon)}\,\nu(dt,dx)\geqq\int_{[0,\infty)\times\mathbb{R}^n} Lu^{(\varepsilon)}\,\mu(dt,dx) \tag{3.8}$$

holds for every $u\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$. Using the Fubini theorem, inequality (3.8) can be rewritten as

$$\int_{[0,\infty)\times\mathbb{R}^n} u\,\nu^{(*\varepsilon)}\,dt\,dx\geqq-\int_{[0,\infty)\times\mathbb{R}^n}(L_\varepsilon u)\,\mu^{(*\varepsilon)}\,dt\,dx. \tag{3.9}$$

Now we show that this inequality holds for $u=u_\varepsilon$.

First we define $u_\varepsilon(0,x)=\lim_{t\downarrow 0}u_\varepsilon(t,x)$ for every $x\in\mathbb{R}^n$ and $u_\varepsilon(t,x)=u_\varepsilon(0,x)$ for $t<0$. $\left(\text{Note that } \lim_{t\downarrow 0}u_\varepsilon(t,x) \text{ exists, since, by the proposition } \frac{\partial}{\partial t}u_\varepsilon(t,x) \text{ is bounded on } (0,\infty)\times\mathbb{R}^n \text{ for every } \varepsilon>0.\right)$ Then taking non-negative functions η, $w\in C_0^\infty(\mathbb{R}^{n+1})$ such that $\int\eta\,dt\,dx=1$ and $w=1$ on $\{(t,x)\in\mathbb{R}^{n+1}\colon |(t,x)|\leqq 1\}$, we define $u_\varepsilon^{(r)}:=u_\varepsilon * h^r$ and $\eta_R(t,x)=\eta(R^{-1}t,R^{-1}x)$. Replacing u in (3.9) by $u_\varepsilon^{(r)}\eta_R$ and then taking $R\to\infty$ we get

$$\int_{[0,\infty)\times\mathbb{R}^n} u_\varepsilon^{(r)}\,\nu^{(*\varepsilon)}\,dt\,dx\geqq-\int_{[0,\infty)\times\mathbb{R}^n}(L_\varepsilon u_\varepsilon^{(r)})\,\mu^{(*\varepsilon)}\,dt\,dx. \tag{3.10}$$

From the proposition it follows that for every $\varepsilon>0$

$$u_\varepsilon^{(r)},\quad \frac{\partial}{\partial t}u_\varepsilon^{(r)}\quad \text{and}\quad \frac{\partial^2}{\partial x^i\,\partial x^j}u_\varepsilon^{(s)}$$

are bounded for every i, j, uniformly in r.

Thus, taking the limit $r\downarrow 0$ in both sides of the inequality (3.10), we obtain (3.9) for $u=u_\varepsilon$ by the Lebesgue convergence theorem, i.e.,

$$\int_{[0,\infty)\times\mathbb{R}^n} u_\varepsilon\,\nu^{(*\varepsilon)}\,dt\,dx \geqq -\int_{[0,\infty)\times\mathbb{R}^n} (L_\varepsilon u_\varepsilon)\,\mu^{(*\varepsilon)}\,dt\,dx.$$

Since $L_\varepsilon u_\varepsilon+f=0$ and by the Ito formula $u_\varepsilon=\mathfrak{R}^\varepsilon f$, this inequality can be rewritten as

$$\int_{[0,\infty)\times\mathbb{R}^n} \mathfrak{R}^\varepsilon f\,\nu^{(*\varepsilon)}\,dt\,dx \geqq \int_{[0,\infty)\times\mathbb{R}^n} f\,u^{(*\varepsilon)}\,dt\,dx.$$

Hence, using the Fubini theorem again

$$\int_{[0,\infty)\times\mathbb{R}^n} (\mathfrak{R}^\varepsilon f)^{(\varepsilon)}\,\nu(dt,dx) \geqq \int_{[0,\infty)\times\mathbb{R}^n} f^{(\varepsilon)}\,\mu(dt,dx)$$

for every non-negative $f\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$.

Step 2. Now in the case of general kernel h we use the same devices as before:

We take the function $\rho(x)=(1+|x|^2)^{-n}(\int(1+|x|^2)^{-n})^{-1}$ and consider the kernels $h^r(t,x)=(h*\rho^r)(t,x)$ where the convolution is understood with respect to the variable x. It is easy to see that h^r has the property (3.7) for every r.

Thus, by virtue of step 1 for every $f\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$ we have the inequality

$$\int_{[0,\infty)\times\mathbb{R}^n} (\mathfrak{R}^{\varepsilon r} f)^{(\varepsilon r)}\,\nu(dt,dx) \geqq \int_{[0,\infty)\times\mathbb{R}^n} f^{(\varepsilon r)}\,\mu(dt,dx) \tag{3.11}$$

for every positive ε and r, where the appearance of the extra index r means that the kernel h^r is taken everywhere instead of h. It remains to take $r\downarrow 0$ in (3.11). It is easy to see that for $r\downarrow 0$

$$\begin{aligned} &(a_{(\varepsilon r)})^{\frac{1}{2}}(t,x)\to(a_{(\varepsilon)})^{\frac{1}{2}}(t,x)\\ &b_{(\varepsilon r)}(t,x)\to b_{(\varepsilon)}(t,x)\\ &c_{(\varepsilon r)}(t,x)\to c_{(\varepsilon)}(t,x)\end{aligned} \tag{3.12}$$

for every $(t,x)\in(0,\infty)\times\mathbb{R}^n$, uniformly on compact subsets. Moreover, $(a_{(\varepsilon)})^{\frac{1}{2}}$, $b_{(\varepsilon)}$ are bounded and satisfy a Lipschitz condition in x on $\{(t,x)\in(0,\infty)\times\mathbb{R}^n\colon |(t,x)|\leqq R\}$ for every R. Therefore, by Theorem 3 from [2], for every $\lambda>0$ and $T\geqq 0$

$$\lim_{r\downarrow 0} P[\sup_{0\leqq t\leqq T} |x_{\varepsilon r}^{sx}(t)-x_\varepsilon^{sx}(t)|\geqq\lambda]=0 \tag{3.13}$$

for every $s\geqq 0$ and $x\in\mathbb{R}^n$. Taking into account that $c_{\varepsilon r}$ is bounded uniformly in r, by using (3.12) and (3.13) we get for every $\varepsilon>0$ that

$$\lim_{r\downarrow 0}(\mathfrak{R}^{\varepsilon r} f)(s,x)=(\mathfrak{R}^\varepsilon f)(s,x) \tag{3.14}$$

for every $(s,x)\in[0,\infty)\times\mathbb{R}^n$, and that for all r

$$|\mathfrak{R}^{\varepsilon r} f| \leqq K \tag{3.15}$$

with a constant K. Further, it is easy to see that for every $\varepsilon>0$

$$\lim_{r\downarrow 0} h^{\varepsilon r}(t,x) = h^{\varepsilon}(t,x) \tag{3.16}$$

$$\lim_{r\downarrow 0} f^{(\varepsilon r)}(t,x) = f^{(\varepsilon)}(t,x) \tag{3.17}$$

for every $(t,x)\in\mathbb{R}^{n+1}$, and for every $(t,x)\in[0,\infty)\times\mathbb{R}^n$ respectively.

Now using (3.14)–(3.17), we obtain (2.8) for every $\varepsilon>0$, by taking $r\downarrow 0$ in both sides of (3.11) and using the Lebesgue convergence theorem. Hence the rest of the lemma follows easily.

4. Coincidence of the One-Dimensional Distributions

Consider the processes

$$\xi(t) = \int_0^t \delta(s)\,dW(s) + \int_0^t \beta(s)\,ds$$

$$\varphi(t) = \int_0^t \gamma(s)\,ds$$

from the introduction. Suppose that $\alpha = \delta\delta^T$ satisfies the condition

$$\sum_{i,j} \alpha^{ij} z^i z^j \geqq p|z|^2 \tag{4.1}$$

for every $(t,\omega)\in[0,\infty)\times\Omega$ and $z\in\mathbb{R}^n$, where p is a fixed positive constant.

Let $\eta(t)=(t,\xi(t))$, and let μ be the Green measure of η with killing rate γ, i.e.

$$\mu(\Gamma) = E\int_0^\infty \chi_\Gamma(\eta(t))\exp(-\varphi(t))\,dt,$$

for every Borel set $\Gamma\subset[0,\infty)\times\mathbb{R}^n$.

Let $f=f(t,\omega)$ be a bounded non-negative measurable process. Then it is easy to see that the measure μ_f defined by

$$\mu_f(\Gamma) = E\int_0^\infty \chi_\Gamma(\eta(t)) f(t)\exp(-\varphi(t))\,dt$$

is absolutely continuous with respect to μ.

Let us define the functions $a=(a^{ij}(t,x))$, $b=(b^i(t,x))$ and $c=c(t,x)$ as follows

$$a^{ij} := \frac{d\mu_\alpha^{ij}}{d\mu}, \quad b^i := \frac{d\mu_\beta^i}{d\mu} \quad \text{and} \quad c = \frac{d\mu_\gamma}{d\mu}.$$

Note that a, b and c are Borel measurable functions on $[0,\infty)\times\mathbb{R}^n$ and if (α,β,γ) take values in a bounded closed convex set $A\subset\mathbb{R}^{n\times n}\times\mathbb{R}^n\times\mathbb{R}$ then a, b, c can be chosen such that $(a(t,x), b(t,x), c(t,x))\in A$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^n$. Thus a, b, c are bounded and

$$\sum_{i,j} a^{ij} z^i z^j \geqq p\,|z|^2 \tag{4.2}$$

for every $(t,x)\in[0,\infty)\times\mathbb{R}^n$ and $z\in\mathbb{R}^n$.

Now let us take the kernel h from Sect. 2 and suppose that $0<\mu^{(*\varepsilon)}<\infty$ on $(0,\infty)\times\mathbb{R}^n$ for every $\varepsilon>0$.

Then defining the functions $a_{(\varepsilon)}=(a^{ij}_{(\varepsilon)})$, $b_{(\varepsilon)}=(b^i_{(\varepsilon)})$ and $c_{(\varepsilon)}$ from a, b and c by the formula (2.7), we state the following

Lemma 4.1. *Let $z^\varepsilon(t)=(u_\varepsilon(t),x_\varepsilon(t))$ be the solution of the equation*

$$\begin{aligned} du_\varepsilon(t)&=dt\\ dx_\varepsilon(t)&=(a_{(\varepsilon)})^{\frac{1}{2}}(u_\varepsilon(t),x_\varepsilon(t)\,dW(t)+b_{(\varepsilon)}(u_\varepsilon(t),x_\varepsilon(t))\,dt\\ z_\varepsilon(0)&=z_0^\varepsilon, \end{aligned} \tag{4.3}$$

where z_0^ε is a random variable in $\mathbb{R}^{n+1}$, independent of the Wiener process W and having the density $h^{*\varepsilon}$. Then for $\varepsilon\downarrow 0$ the Green measure of the process z_ε (with killing rate $c_{(\varepsilon)}(z_\varepsilon(t))$) converge weakly to the Green measure μ.

Proof. Let ν be the measure defined by $\nu(\Gamma)=\chi_\Gamma(0)$ for every Borel set $\Gamma\subset\mathbb{R}^{n+1}$, where 0 is the null-vector of $\mathbb{R}^{n+1}$.

Applying the Ito formula for $u(t,\xi(t))$ we have

$$\begin{aligned} \int u\,\nu(dx,dx)=u(0)&=-E\int_0^\infty\left\{\frac{\partial}{\partial t}u(t,\xi(t))+\frac{1}{2}\sum_{i,j}\alpha^{ij}(t)\frac{\partial^2}{\partial x^i\,\partial x^j}u(t,\xi(t))\right.\\ &\qquad\left.+\sum_i\beta^i(t)\frac{\partial}{\partial x^i}u(t,\xi(t))-\gamma(t)\,u(t,\xi)\right\}\exp(\varphi(t))\,dt\\ &=-\int_{[0,\infty)\times\mathbb{R}^n}\frac{\partial}{\partial t}u(t,x)\,\mu(dt,dx)+\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x^i\,\partial x^j}u(t,x)\,\mu^{ij}_\alpha(dt,dx)\\ &\qquad+\sum_i\frac{\partial}{\partial x^i}u(t,x)\,\mu^i_\beta(dt,dx)-u(t,x)\,\mu_\gamma(dt,dx)\\ &=-\int_{[0,\infty)\times\mathbb{R}^n}Lu(t,x)\,\mu(dt,dx) \end{aligned}$$

for every $u\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$. Hence by Lemma 2.2

$$\int_{[0,\infty)\times\mathbb{R}^n} f(t,x)\,\mu(dt,dx)=\lim_{\varepsilon\downarrow 0}(\mathfrak{R}^\varepsilon f)^{(\varepsilon)}(0)$$

holds for every $f\in C_0^\infty([0,\infty)\times\mathbb{R}^n)$.

We can now finish the proof by noticing that $(\mathfrak{R}^\varepsilon f)^{(\varepsilon)}$ is just the integral of f on $[0,\infty)\times\mathbb{R}^n$ with respect to the Green measure of $z_\varepsilon(t)$.

From now on we take $h(t,x)=\psi(t)\,\rho(x)$. Our theorem will be proved after some propositions.

Proposition 4.2. *The Green measure μ is equivalent to the Lebesgue measure m on $[0, \infty) \times \mathbb{R}^n$, i.e.,*

(i) $\mu \ll m$

(ii) $m \ll \mu$.

Proof. The assertion (i) is well-known (see [5]). We sketch the proof of (ii). We can see that for all $\varepsilon > 0$

$$\sum_{i,j} |a^{ij}_{(\varepsilon)}| + \sum_i |b^i_{(\varepsilon)}| + c_{(\varepsilon)} \leqq K \tag{4.4}$$

on $(0, \infty) \times \mathbb{R}^n$, where K is some constant, which does not depend on ε. Moreover from (4.2) it follows that for all $\varepsilon > 0$

$$\sum_{i,j} a^{ij}_{(\varepsilon)}(t, x)\, z^i z^j \geqq p\, |z|^2 \tag{4.5}$$

for every $(t, x) \in (0, \infty) \times \mathbb{R}^n$ and $z \in \mathbb{R}^n$. Thus we can apply Corollary IV.1.8 from [4], which states the following: Suppose that for every $\varepsilon > 0$ we have a non-negative function $f_\varepsilon \in C_0^\infty((0, \infty) \times \mathbb{R}^n)$ such that $\lim_{\varepsilon \downarrow 0}(t_\varepsilon, x_\varepsilon) = (t_0, x_0) \in (0, \infty) \in \mathbb{R}^n$ implies $\lim_{\varepsilon \downarrow 0} u_\varepsilon(t_\varepsilon, x_\varepsilon) = 0$, where $u_\varepsilon \in C_b^{1\,2}((0, \infty) \times \mathbb{R}^n)$ is the solution of $L_\varepsilon u_\varepsilon + f_\varepsilon = 0$. Then for every $\lambda > 0$

$$\lim_{\varepsilon \downarrow 0} m(\{(t, x) : f_\varepsilon \geqq \lambda\} \cap Q_{TR}) = 0$$

for every $T \geqq t_0$ and $R > 0$, where $Q_{TR} = (t_0, T) \times \{x \in \mathbb{R}^n : |x| \leqq R\}$.

Combining this statement with Lemma 2.2 we get assertion (ii).

Proposition 4.3. *For $\varepsilon \downarrow 0$*

$$a_{(\varepsilon)}(t, x) \to a(t, x)$$

$$b_{(\varepsilon)}(t, x) \to b(t, x)$$

$$c_{(\varepsilon)}(t, x) \to c(t, x)$$

for m-almost all $(t, x) \in [0, \infty] \times \mathfrak{R}^n$.

Proof. Let $\dfrac{d\mu}{dm}$ be denoted by g. It is not difficult to show that for $\varepsilon \downarrow 0$

$$(a^{ij}\mu)^{(*\varepsilon)} \to a^{ij}(t, x)\, g(t, x)$$

$$\mu^{(*\varepsilon)}(t, x) \to g(t, x)$$

for m-almost all $(t, x) \in [0, \infty) \times \mathbb{R}^n$.

Consequently,

$$\lim_{\varepsilon \downarrow 0} a^{ij}_{(\varepsilon)}(t, x) = \lim_{\varepsilon \downarrow 0} (a^{ij}\mu)^{(*\varepsilon)} \lim_{\varepsilon \downarrow 0} (\mu^{(*\varepsilon)})^{-1} = a^{ij}(t, x)$$

for m-almost all $(t, x) \in [0, \infty) \times \mathbb{R}^n$. The other assertions can be shown in the same way.

Proposition 4.4. *Suppose $\gamma(t)$ is a deterministic process (i.e. it does not depend on $\omega\in\Omega$). Then*

$$a^{ij}(t,x)=E(\alpha^{ij}(t)\,|\,\xi(t)=x)$$

$$b^{i}(t,x)=E(\beta^{i}(t)\,|\,\xi(t)=x)$$

$$c(t,x)=E(\gamma(t)\,|\,\xi(t)=x)$$

for m-almost every $(t,x)\in[0,\infty)\times\mathbb{R}^n$.

Proof. Recall the definition of $E(\alpha^{ij}(t)\,|\,\xi(t)=x)$ via the Radon-Nikodym theorem:

One considers the measure Q_t defined by the formula

$$Q_t(F)=\int_{\Omega}\chi_F(\xi(t))\,\alpha^{ij}(t)\,P(d\omega)$$

for every Borel set $F\subset\mathbb{R}^n$. It is obvious that $Q_t\ll P_t$, where P_t is the distribution of $\xi(t)$ on $\mathbb{R}^n$. Therefore there exists a function $q(t,x)$, such that $q(t,\cdot)$ is Borel measurable and for every $t\in[0,\infty)$

$$Q_t(F)=\int_{\mathbb{R}}\chi_\Gamma(x)\,q(t,x)\,P_t(dx)$$

holds for every Borel set F. For every t the function $q(t,\cdot)$ is unique up to a set of P_t-measure 0. Since $\xi(t,\omega)$ and $\alpha^{ij}(t)$ are measurable in (t,ω), one can show that $q(t,x)$ can be chosen to be Borel measurable in (t,x). We define $E(\alpha^{ij}(t)\,|\,\xi(t)=x)=q(t,x)$. It is easy to see that for every t $E(\alpha^{ij}\,|\,\xi(t))=q(t,\xi(t))$ P-almost surely.

By the Fubini theorem

$$\mu_\alpha^{ij}=\int_0^\infty E\{\chi_\Gamma(t,\xi(t))\exp(\varphi(t))\,\alpha^{ij}(t)\}\,dt$$

$$=E\int_0^\infty \chi_\Gamma(t,\xi(t))\exp(-\varphi(t))\,E(\alpha^{ij}(t)\,|\,\xi(t))\,dt.$$

Hence it follows that $\dfrac{d\mu_\alpha^{ij}}{d\mu}=E(\alpha^{ij}(t)\,|\,\xi(t)=x)$ for $\mu\sim m$-almost every $(t,x)\in[0,\infty)\times\mathbb{R}^n$. Similarly we get that $b^i(t,x)=E(\beta^i(t)\,|\,\beta(t)=x)$ and $c(t,x)=E(\gamma(t)\,|\,\beta(t)=x)$.

Proposition 4.5. *For a sequence $\varepsilon_n\downarrow 0$ the distribution of the process z_ε from Lemma 4.1 converges weakly on $C[0,T]$ to the distribution of the process $\bar{z}(t)=(t,\bar{x}(t))$ for every $T\geqq 0$, where $\bar{x}(t)$ is a weak solution of the equation*

$$d\bar{x}(t)=a^{\frac{1}{2}}(t,\bar{x}(t))\,dW(t)+b(t,\bar{x}(t))\,dt$$

with initial condition $\bar{x}(0)=0\in\mathbb{R}^n$.

Proof. Taking into account (4.4), (4.5) and Proposition 4.3, we can prove this proposition in the same way as Theorem II.6.1 in [5].

Now we prove our main result which we stated in the introduction.

Theorem 4.6. *Under the condition* (4.1) *there exist bounded measurable functions* $\sigma\colon [0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ *and* $b\colon [0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ *such that* $\sigma(t,x)=\{E(\delta(t)\delta^T(t)|\xi(t)=x)\}^{\frac{1}{2}}$, $b(t,x):=E(\beta(t)|\xi(t)=x)$ *m-almost all* $(t,x)\in[0,\infty)\times\mathbb{R}^n$, *and the stochastic differential equation*

$$dx(t)=\sigma(t,x(t))\,dW(t)+b(t,x(t))\,dt$$
$$x(0)=0$$

admits a weak solution $\bar{x}(t)$ *which has the same one-dimensional distributions as* $\xi(t)$.

Proof. From Lemma 4.1 and Proposition 4.5 we get that the Green measure of $(t,\xi(t))$ (with killing rate γ) is identical to the Green measure of $(t,\bar{x}(t))$ (with killing rate $c(t,\bar{x}(t))$). Let us take $\gamma(t)\equiv 1$. Then we have $c(t,x)=\gamma=1$ and

$$E\int_0^\infty e^{-t}f(t,\xi(t))\,dt=E\int_0^\infty e^{-t}f(t,\bar{x}(t))\,dt$$

for every bounded non-negative Borel measurable function f. Taking $f(t,x)=e^{-\lambda t}g(x)$ with arbitrary non-negative constant λ and functions $g\in C_0(\mathbb{R}^n)$ we get

$$\int_0^\infty e^{-\lambda t}e^{-t}Eg(\xi(t))\,dt=\int_0^\infty e^{-\lambda t}e^{-t}Eg(\bar{x}(t))\,dt \tag{4.6}$$

for every $\lambda\geqq 0$ and $g\in C_0(\mathbb{R}^n)$.

Since $Eg(\bar{x}(t))$ and $Eg(\xi(t))$ are continuous in t, from (4.6) we get that for every $t\geqq 0$

$$Eg(\bar{x}(t))=Eg(\xi(t))$$

for every $g\in C_0(\mathbb{R}^n)$. Hence it follows that the distributions of $\xi(t)$ and $\bar{x}(t)$ are the same for every $t\geqq 0$.

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